

Closing Thur: 14.7
 Closing Tue: 15.1, 15.2

Entry Task

(Two Quick Calc 1 Questions)

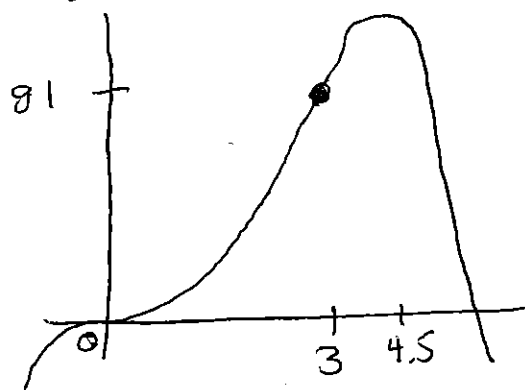
(a) Find the global max/min of

$$z = g(x) = 6x^3 - x^4$$

on the interval $0 \leq x \leq 3$

$$\begin{aligned} \text{I) } z' &= 18x^2 - 4x^3 \stackrel{?}{=} 0 && \left. \begin{array}{l} \text{NOT} \\ \text{IN} \\ \text{REGION!} \end{array} \right\} \\ &2x^2(9 - 2x) \stackrel{?}{=} 0 \\ &\Rightarrow x=0 \quad \text{or} \quad x = \frac{9}{2} = 4.5 \\ &\text{CRITICAL NUMBERS} \end{aligned}$$

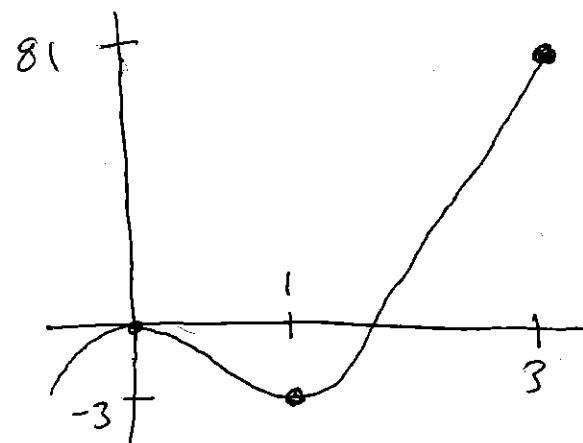
$$\text{II) } \begin{cases} g(0) = 6(0)^3 - (0)^4 = 0 \leftarrow \text{ABS MIN} \\ g(3) = 6(3)^3 - (3)^4 = 81 \leftarrow \text{ABS MAX} \end{cases}$$



(b) Find the global max/min of
 $z = h(x) = 6x^3 - 9x^2$
 on the interval $0 \leq x \leq 3$

$$\begin{aligned} \text{I) } z' &= 18x^2 - 18x \stackrel{?}{=} 0 && \left. \begin{array}{l} \text{IS} \\ \text{IN} \\ \text{REGION} \end{array} \right\} \\ &18x(x-1) \stackrel{?}{=} 0 \\ &\Rightarrow x=0 \quad \text{or} \quad x=1 \end{aligned}$$

$$\begin{aligned} \text{II) } & \begin{aligned} & \rightarrow h(0) = 6(0)^3 - 9(0)^2 = 0 \\ \text{ENDPOINTS } & h(1) = 6(1)^3 - 9(1)^2 = -3 \leftarrow \text{ABS. MIN} \\ & \rightarrow h(3) = 6(3)^3 - 9(3)^2 = 81 \leftarrow \text{ABS. MAX} \end{aligned} \end{aligned}$$



14.7 Max/Min *continued*

Absolute (global) max/min method

Step 1: Critical pts inside region?

Step 2: Max/Min over each boundary?

Step 3: Function value at all these pts.

Biggest output = global max

Smallest output = global min

See last lecture notes for two full examples (review sheet for another)

Yet another example:

Find the absolute max/min of

$$z = f(x, y) = 6x^3 - y^2$$

over the region bounded by

$$y = x^2 \text{ and } y = 3x.$$

$$\text{I) } f_x = 18x^2 \stackrel{?}{=} 0 \Rightarrow x=0$$

$$f_y = -2y \stackrel{?}{=} 0 \Rightarrow y=0$$

$$(x, y) = (0, 0) \text{ ONLY CRITICAL PT.}$$

$$\text{II) [A] } y = x^2, \quad 0 \leq x \leq 3$$

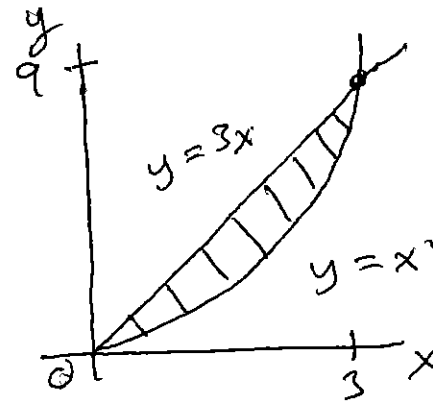
$$\Rightarrow z = f(x, x^2) = 6x^3 - (x^2)^2 = 6x^3 - x^4$$

● SEE PREVIOUS PAGE!

$$\text{[B] } y = 3x, \quad 0 \leq x \leq 3$$

$$\Rightarrow z = f(x, 3x) = 6x^3 - (3x)^2 = 6x^3 - 9x^2$$

● SEE PREVIOUS PAGE



Homework hints

In applied optimization problems,

- Identify what you are optimizing!
- Label Everything.
- Identify given facts (constraints)
- Use the constraints and labels to give a 2 variable function for the objective.

HW Examples:

- Find the points on the cone $z^2 = x^2 + y^2$ that are closest to $(4, 2, 0)$.

Objective: Minimize **distance** from (x, y, z) points on the cone to the point $(4, 2, 0)$ given that $z^2 = x^2 + y^2$.

$$\text{DIST TO } (4, 2, 0) = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$$

$$\text{CONSTRAINT: } z^2 = x^2 + y^2$$

$$\Rightarrow D(x, y) = \sqrt{(x-4)^2 + (y-2)^2 + x^2 + y^2}$$

Now

$$D_x = \frac{1}{2\sqrt{\quad}} (2(x-4) + 2x) \stackrel{?}{=} 0$$

$$D_y = \frac{1}{2\sqrt{\quad}} (2(y-2) + 2y) \stackrel{?}{=} 0$$

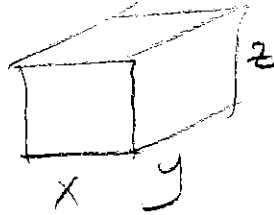
$$4x - 8 \stackrel{?}{=} 0 \Rightarrow x = 2$$

$$4y - 4 \stackrel{?}{=} 0 \Rightarrow y = 1$$

⋮

2. Find the dimensions of the box with volume 1000 cm^3 that has minimum surface area.

Objective: Minimize **surface area** given that volume is 1000.



$$\text{Surface Area} = 2xy + 2yz + 2xz$$

$$\text{CONSTRAINT: } xyz = 1000 \Rightarrow z = \frac{1000}{xy}$$

$$S(x,y) = 2xy + 2y \frac{1000}{xy} + 2x \frac{1000}{xy}$$

$$S(x,y) = 2xy + \frac{2000}{x} + \frac{2000}{y}$$

$$S_x = 2y - \frac{2000}{x^2} \stackrel{?}{=} 0$$

$$S_y = 2x - \frac{2000}{y^2} \stackrel{?}{=} 0$$

$$\Rightarrow y = \frac{1000}{x^2}$$

$$2x - \frac{2000}{\left(\frac{1000}{x^2}\right)^2} = 0$$

$$2x \left(\frac{(1000)^2}{x^4}\right) - 2000 = 0$$

$$\frac{(1000)^2}{x^3} = 1000$$

$$x^3 = 1000$$

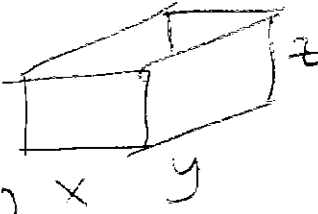
$$x = (1000)^{1/3}$$

3. You want to build aquariums with slate for the base and glass for the sides (and no top).

Assume slate costs \$5 per in² and glass costs \$1 per in².

If the volume must be 1000 in³, then what dimensions will minimize cost?

Objective: Minimize **cost** when volume needs to be 1000.



COST PER in^2
 \downarrow
 AREA OF BASE
 \downarrow
 $\text{COST} = 5xy + 1(2xz + 2yz)$

COST PER in^2
 \downarrow
 AREA OF GLASS
 \downarrow
 $xyz = 1000 \Rightarrow z = \frac{1000}{xy}$

$$C(x,y) = 5xy + 2x \frac{1000}{xy} + 2y \frac{1000}{xy}$$

$$C(x,y) = 5xy + \frac{2000}{y} + \frac{2000}{x}$$

$$C_x = 5y - \frac{2000}{x^2} \stackrel{?}{=} 0$$

$$C_y = 5x - \frac{2000}{y^2} \stackrel{?}{=} 0$$

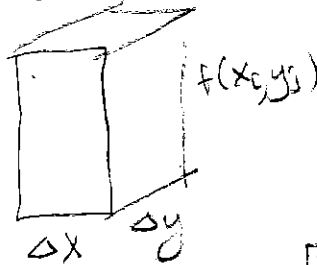
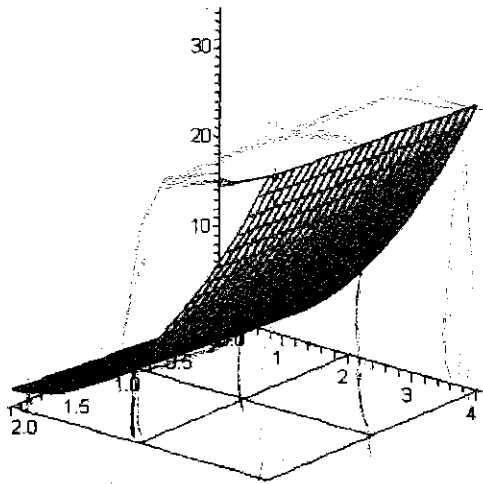
Close today: 14.7
 Close Tue: 15.1, 15.2
 Close Thur: 15.3
 Office Hours Today: 12:30-2:30pm (MSC)

15.1-15.3 Intro to double Integrals

Goal: Give a definition for the volume between a *given surface* and a *given region* on the *xy*-plane.

In all of ch. 15, you are given two things:

1. A surface: $z = f(x,y)$
2. A region drawn on the xy -plane.



Example:

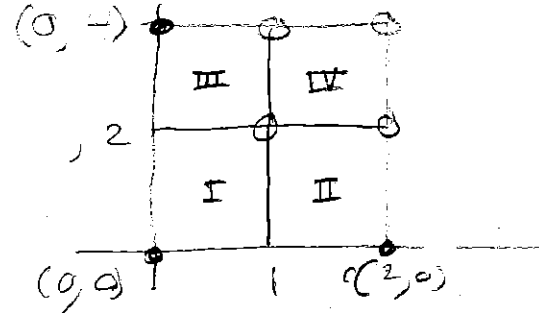
The volume under

$$z = f(x,y) = x + 2y^2$$

and above

$$R = [0,2] \times [0,4] = \{(x,y) : 0 \leq x \leq 2, 0 \leq y \leq 4\}$$

- (a) Break the region R into $m = 2$ columns and $n = 2$ rows; 4 sub-regions;
- (a) Approx. using a rectangular box over each region (use *upper-right* endpoints).



$$\Delta x = \frac{2-0}{2} = 1, \quad \Delta y = \frac{4-0}{2} = 2$$

$$\Delta A = \Delta x \Delta y = (1)(2) = \text{AREA OF "BASE"}$$

- I $f(1,2) = (1) + 2(2)^2 = 9 \Rightarrow \text{VOL} = f(1,2) \Delta A = 9 \cdot 2 = 18$
- II $f(2,2) = (2) + 2(2)^2 = 10 \Rightarrow \text{VOL} = f(2,2) \Delta A = 10 \cdot 2 = 20$
- III $f(1,4) = (1) + 2(4)^2 = 33 \Rightarrow \text{VOL} = f(1,4) \Delta A = 33 \cdot 2 = 66$
- IV $f(2,4) = (2) + 2(4)^2 = 34 \Rightarrow \text{VOL} = f(2,4) \Delta A = 34 \cdot 2 = 68$

$$\text{TOTAL VOL} \approx 18 + 20 + 66 + 68 = 172 \quad \leftarrow \text{BIG OVER APPROXIMATION}$$

Formally, we define:

$$\iint_R f(x, y) dA = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}, y_{ij}) \Delta A$$

= the 'signed' volume between $f(x, y)$ and the xy -plane over R .

If $f(x, y)$ is above the xy -plane it is positive.

If $f(x, y)$ is below the xy -plane it is negative.

General Notes and Observations:

$z = f(x, y)$ = height on surface

R = the region on the xy -plane

ΔA = area of base = $\Delta x \Delta y = \Delta y \Delta x$

$f(x_{ij}, y_{ij}) \Delta A$ = (height)(area of base)

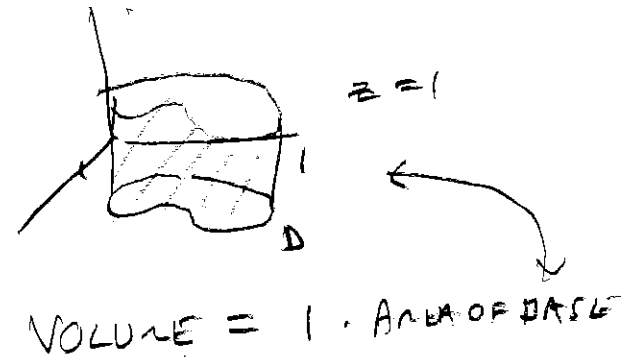
= volume of one approximating box

Units of $\iint_R f(x, y) dA$ are

(units of $f(x, y)$)(units of x)(units of y)

Quick application note:

$$\iint_R 1 dA = \text{"Area of } R", \text{ and}$$

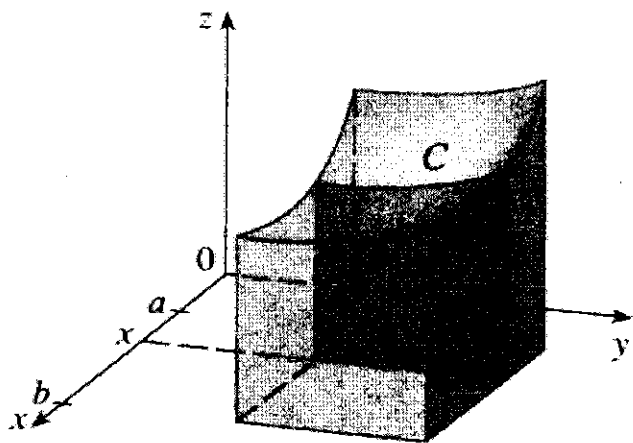


VOLUME = 1 * AREA OF BASE

Iterated Integrals

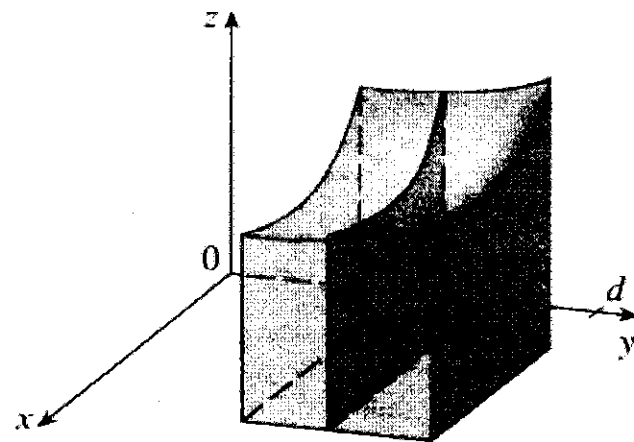
If you fix x : The area under this curve is

$$\int_c^d f(x, y) dy = \text{"cross sectional area under the surface at this fixed } x \text{ value"}$$



If you fix y : The area under this curve

$$\int_a^b f(x, y) dx = \text{"cross sectional area under the surface at this fixed } y \text{ value"}$$



From Math 125,

$$\text{Vol} = \int_a^b \text{Area}(x) dx = \int_a^b \left(\int_c^d f(x, y) dy \right) dx$$

$$\text{Vol} = \int_c^d \text{Area}(y) dy = \int_c^d \left(\int_a^b f(x, y) dx \right) dy$$

Quick Example: Evaluate

$$(a) \int_2^6 \left(\int_1^8 y \, dx \right) dy = \int_2^6 \left(y \times \left. \vphantom{y} \right|_{x=1}^{x=8} \right) dy$$

$$= \int_2^6 8y - y \, dy = \int_2^6 7y \, dy$$

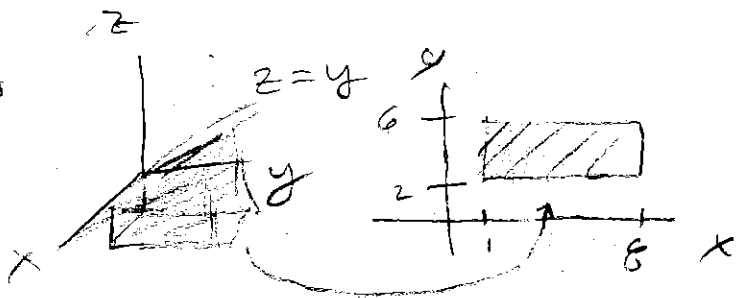
CROSS-SECTIONAL AREA WITH y IS FIXED

$$= \frac{7}{2} y^2 \Big|_2^6 = \frac{7}{2} (6^2 - 2^2)$$

$$= \frac{7}{2} (36 - 4) = \frac{7}{2} \cdot 32 = 2 \cdot 16$$

$$= \boxed{112}$$

$z=y$ IS A PLANE



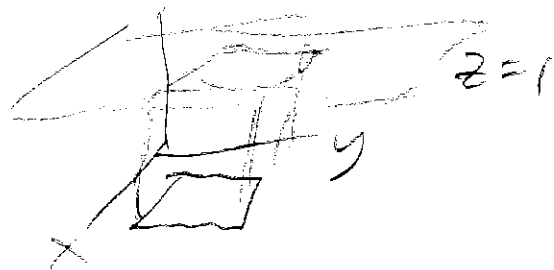
$$(b) \int_2^6 \left(\int_1^8 1 \, dx \right) dy = \int_2^6 \left(x \Big|_1^8 \right) dy$$

$$= \int_2^6 8 - 1 \, dy = \int_2^6 7 \, dy$$

CROSS-SECTIONAL AREA WITH y IS FIXED

$$= 7y \Big|_2^6 = 7(6-2)$$

$$= \boxed{28} = \text{Area of Region}$$



Examples (like 15.2 HW):

1. Find the volume under

$$z = x + 2y^2 \text{ and}$$

$$\text{above } 0 \leq x \leq 2, \quad 0 \leq y \leq 4$$

$$\int_0^4 \left(\int_0^2 x + 2y^2 dx \right) dy$$

$$\int_0^4 \left(\frac{1}{2}x^2 + 2y^2x \Big|_0^2 \right) dy$$

$$\int_0^4 2 + 4y^2 dy$$

$$2y + \frac{4}{3}y^3 \Big|_0^4$$

$$8 + \frac{4}{3} \cdot 64 = 8 + \frac{256}{3}$$

$$= 93.\bar{3}$$

OR

$$\int_0^2 \left(\int_0^4 x + 2y^2 dy \right) dx$$

$$\int_0^2 \left(xy + \frac{2}{3}y^3 \Big|_0^4 \right) dx$$

$$\int_0^2 4x + \frac{128}{3} dx$$

$$2x^2 + \frac{128}{3}x \Big|_0^2$$

$$8 + \frac{256}{3}$$

$$= 93.\bar{3}$$

COMPARE TO OUR APPROXIMATION

EARLIER

$$2. \int_0^3 \left(\int_0^1 2xy \sqrt{x^2 + y^2} dx \right) dy$$

$$2x \sqrt{x^2 + 1}$$

SUBSTITUTION!!!

$$\int_0^3 \left(\int_{y^2}^{1+y^2} 2xy \sqrt{u} \frac{1}{2x} du \right) dy$$

$$u = x^2 + y^2$$

$$du = 2x dx$$

$$\frac{1}{2x} du = dx$$

$$\int_0^3 \left(y \frac{2}{3} u^{3/2} \Big|_{y^2}^{1+y^2} \right) dy$$

$$\int_0^3 \frac{2}{3} y (1+y^2)^{3/2} - \frac{2}{3} y (y^2)^{3/2} dy$$

$$\int_0^3 \frac{2}{3} y (1+y^2)^{3/2} - \frac{2}{3} y^4 dy$$

$$\int_0^3 \frac{2}{3} y (1+y^2)^{3/2} dy$$

$$u = 1+y^2$$

$$du = 2y dy$$

$$\frac{1}{y} du = 2 dy$$

$$\int_1^{10} \frac{2}{3} y u^{3/2} \frac{1}{2y} du$$

$$\frac{1}{3} \frac{2}{5} u^{5/2} \Big|_1^{10} = \frac{2}{15} (10^{5/2} - 1)$$

$$-\frac{2}{15} y^5 \Big|_0^3$$

$$-\frac{2}{15} (3)^5 = -\frac{162}{5}$$

$$\frac{2}{15} (10^{5/2} - 1) - \frac{162}{5}$$

$$\approx 9.63$$

3. Find the double integral of

$$f(x, y) = y \cos(x + y)$$

over the rectangular region

$$0 \leq x \leq \pi, \quad 0 \leq y \leq \pi/2$$

$$\int_0^{\pi/2} \left[\int_0^{\pi} y \cos(x+y) dx \right] dy$$

OR $\int_0^{\pi} \left(\int_0^{\pi/2} y \cos(x+y) dy \right) dx$

$$\int_0^{\pi/2} \left[y \sin(x+y) \Big|_0^{\pi} \right] dy$$

$$\int_0^{\pi/2} \left[y \sin(\pi+y) - y \sin(y) \right] dy$$

$$\int_0^{\pi/2} \left[y (\sin(\pi+y) - \sin(y)) \right] dy \quad \text{By PARTS!}$$

$$u = y \quad \begin{aligned} dv &= \sin(\pi+y) - \sin(y) dy \\ du &= dy \quad v &= -\cos(\pi+y) + \cos(y) \end{aligned}$$

$$y (-\cos(\pi+y) + \cos(y)) \Big|_0^{\pi/2} - \int_0^{\pi/2} (-\cos(\pi+y) + \cos(y)) dy$$

$$\frac{\pi}{2} \left[-\cos\left(\frac{3\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) \right] - 0 \left[-\cos(\pi) + \cos(0) \right] - \left[-\sin(\pi+y) + \sin(y) \Big|_0^{\pi/2} \right]$$

$$\frac{\pi}{2} (0+0) - 0(1+1) = - \left[(-\sin(\frac{\pi}{2}) + \sin(\pi)) - (-\sin(0) + \sin(0)) \right]$$

$$= 0 - [(-(-1)+1) - 0] = \boxed{-2}$$

15.2 Double Integrals over General Regions

For the rectangular region, R , given by

$$a \leq x \leq b, \quad c \leq y \leq d$$

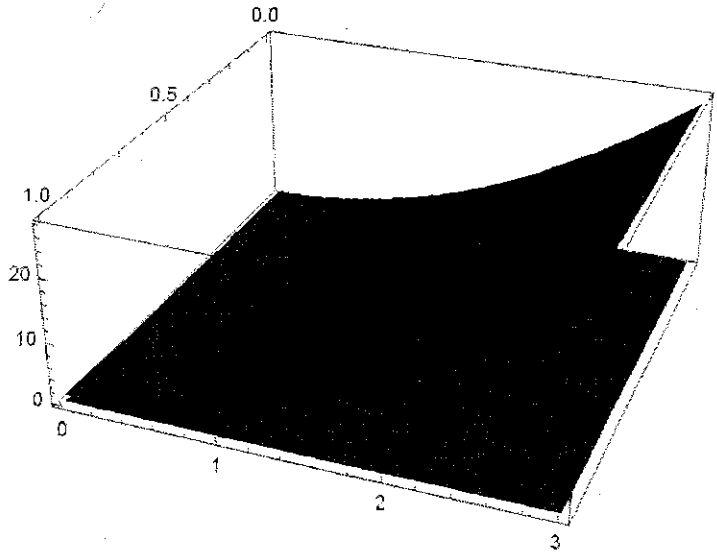
we learned:

$$\begin{aligned} \iint_R f(x, y) dA &= \int_a^b \left(\int_c^d f(x, y) dy \right) dx \\ &= \int_c^d \left(\int_a^b f(x, y) dx \right) dy \end{aligned}$$

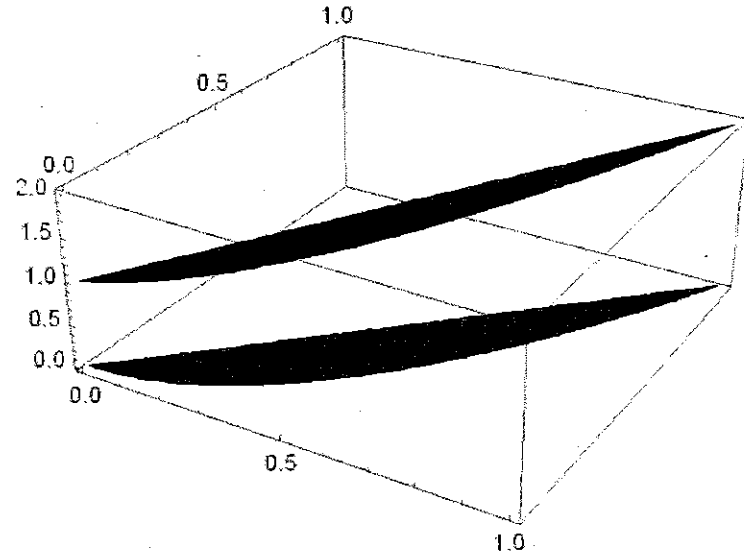
In 15.2, we discuss regions, R , other than rectangles.

Type 1 Regions (Top/Bot)	Type 2 Regions (Left/Right)
<p>Given x in the range, $a \leq x \leq b$, we have $g_1(x) \leq y \leq g_2(x)$</p>	<p>Given y in the range, $c \leq y \leq d$, we have $h_1(y) \leq x \leq h_2(y)$</p>
$\int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x, y) dy \right) dx$	$\int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right) dy$

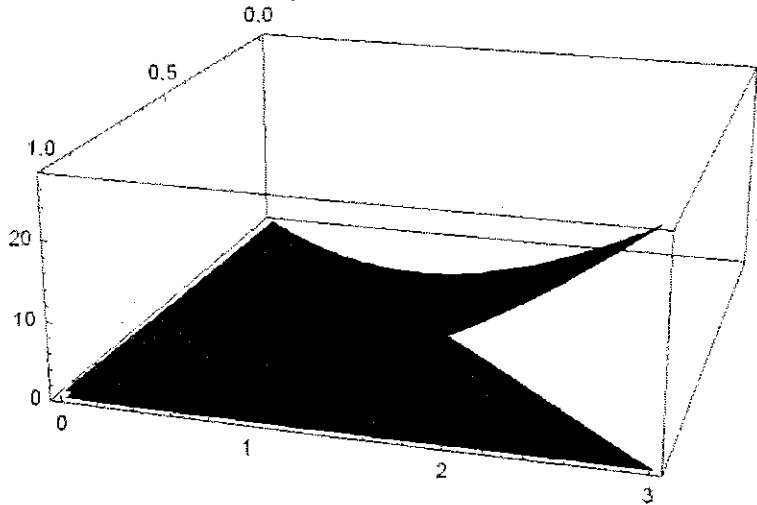
The surface $z = x + 3y^2$ over the rectangular region $R = [0,1] \times [0,3]$



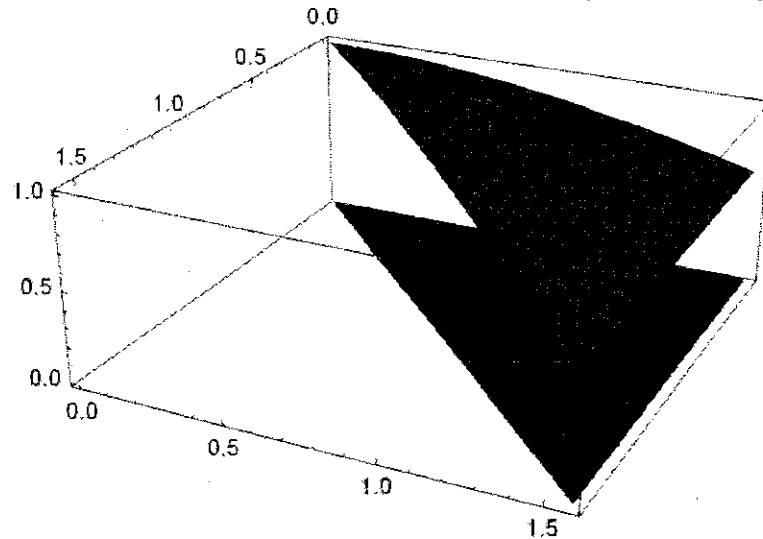
The surface $z = x + 1$ over the region bounded by $y = x$ and $y = x^2$.



The surface $z = x + 3y^2$ over the triangular region with corners $(x,y) = (0,0)$, $(1,0)$, and $(1,3)$.



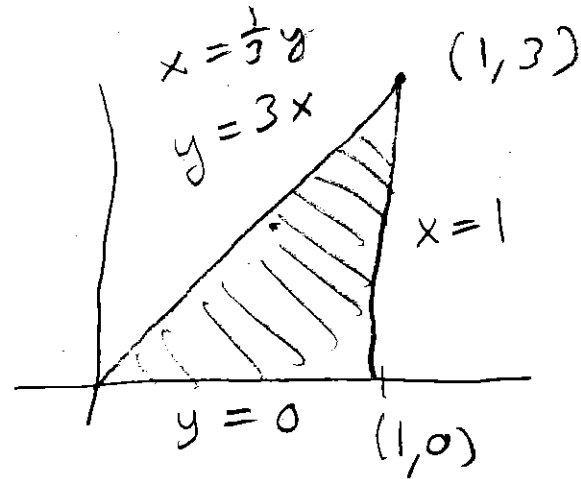
The surface $z = \sin(y)/y$ over the triangular region with corners at $(0,0)$, $(0, \pi/2)$, $(\pi/2, \pi/2)$.



Examples:

1. Let D be the triangular region in the xy -plane with corners $(0,0)$, $(1,0)$, $(1,3)$.

Evaluate $\iint_D x + 3y^2 dA$



OPTION 1: FIX x
For some $0 \leq x \leq 1$

$$\Rightarrow 0 \leq y \leq 3x$$

$y=0$ IS ALWAYS THE "BOTTOM" BOUND
 $y=3x$ IS ALWAYS THE "TOP" BOUND

$$\int_0^1 \left(\int_0^{3x} x + 3y^2 dy \right) dx$$

OPTION 2: Fix y
for some $0 \leq y \leq 3$

$$\Rightarrow \frac{1}{3}y \leq x \leq 1$$

$x = \frac{1}{3}y$ IS ALWAYS THE "LEFT" BOUND
 $x = 1$ IS ALWAYS THE "RIGHT" BOUND

$$\int_0^3 \left(\int_{\frac{1}{3}y}^1 x + 3y^2 dx \right) dy$$

OR